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### Universality change in stochastic cellular automaton with applied site exchange

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Extended mean-field theory is used for studying the effect of nonlocal site exchange on the critical transition of the one-dimensional, stochastic cellular automaton of Rule 18 defined by Wolfram [Rev. Mod. Phys. **55**, 601 (1993)]. The analysis is carried out at different levels taking  $n$ -point correlations explicitly into consideration. Extrapolation to the  $n \rightarrow \infty$  (exact) limit gives results, which agree well with simulation. By applying nonlocal site exchange mixing the universality of the transition changes and a continuous crossover of critical point and  $\beta$  exponent is observed.

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Cellular automaton (CA) models have great importance in many fields of science such as physics, chemistry, biology, and computer science [1]. They are used to model other complex nonequilibrium systems of other branches of science too. For example, the dynamics of traffic flow, forest fires, or economical systems can also be studied with CA [2].

Stochastic CA may exhibit phase transition even in one dimension unlike equilibrium statistical physics systems [1]. Mean-field theory—as usual—can well describe transitions with small fluctuations and with exponentially decreasing correlations only. It works well for higher-dimensional systems (exact in six dimensions) and for long range interaction models. An extended mean-field solution technique, proposed and tested on CA [5], has shown that the critical behavior can be well described on applying appropriate extrapolation to the results. This method predicts the location of the phase transition point and the order parameter exponent with reasonable accuracy.

Site exchange introduced to CA models by [3] can create long range interactions, destroying the correlations generated by the CA rule. This kind of sequential mix-

ing process—which models short or long range moves in epidemiology—is introduced between subsequent CA state updates and does not change the number of living cells. If the degree of mixing  $m$  tends to infinity the steady state concentration of living cells can be well approximated by simple mean-field calculation. However, for small values of  $m$  the correlations are not completely destroyed and this approximation usually gives incorrect results. One expects that the mixing modifies the phase transition features as well, toward a transition well described by the mean-field solution. Sometimes it changes the order of transition as well [4].

In this paper we extend our previous analysis of the stochastic Rule 18 [5] with the site-exchange rule and determine the change of universality of the critical transition as a function of the strength of this mixing. Universality classes for critical nonequilibrium systems are far less understood than for equilibrium systems. There exists a hypothesis that models with a scalar order-parameter and short-range interactions exhibiting a continuous transition to an absorbing state belong to the universality class of directed percolation (DP) or Reggeon field theory [6]. We keep track of the universality by de-

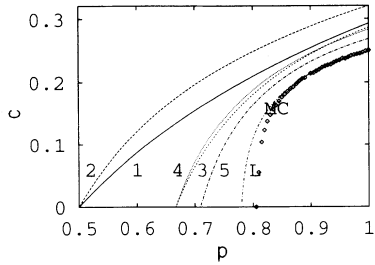


FIG. 1. Average concentration vs  $p$  for the levels of one-, two-, three-, four-, and five-point correlation approximations and the limit curve  $L$ . Simulations ( $\diamond$ ) are performed on  $L = 40\,000$  system.

termining the order parameter exponent  $\beta$  :

$$c(p, m) \sim (p - p_c)^{\beta(m)} \quad (1)$$

as a function of nonlocal site-exchange strength  $m$ . We

$$s(t+1, j) = \begin{cases} X & \text{if } s(t, j-1) = 0, \quad s(t, j) = 0, \quad s(t, j+1) = 1, \\ X & \text{if } s(t, j-1) = 1, \quad s(t, j) = 0, \quad s(t, j+1) = 0, \\ 0 & \text{otherwise,} \end{cases}$$

where  $X \in \{0, 1\}$  is a two valued random variable such that

$$\text{Prob}(X = 1) = p \quad .$$

This update is followed by a sequential rule:  $m \times$  (number of living cells) (1) are selected randomly and swapped with other site values (either zeros or ones) selected at random from all over the lattice.

The  $p = 1$  (deterministic),  $m = 0$  limit is equivalent to the Rule 18 CA defined by Wolfram [1]. The mixing-free case has been investigated by [5,7,8] and has been found to exhibit a second order phase transition at  $p_c = 0.8086(\pm 0.0002)$ , belonging to the DP universality class [9]. In contrast, the simple mean-field theory predicts a continuous transition characterized by  $p_c = 0.5$  and  $\beta = 1$  [1].

As it was described in [5] we extend the usual mean-field calculation by setting up steady state equations for  $n$ -point configuration probabilities :

$$P_n^{t+1}(s_1, \dots, s_n) = f(\{P_m^t(s_1, \dots, s_n)\}) = P_n^t(s_1, \dots, s_n), \quad (2)$$

where the function  $f$  is determined by the rule dependent on  $\{P_m^t\}$  "block probabilities" at time  $t$ .

At the level of  $k$ -point approximation the correlations are neglected for  $n > k$ , that is,  $P_n(s_1, \dots, s_n)$  is expressed by using the Bayesian extension process [10,11],

$$P_n(s_1, \dots, s_n) = \frac{\prod_{j=0}^{j=n-k} P_k(s_{1+j}, \dots, s_{k+j})}{\prod_{j=1} P_{k-1}(s_{1+j}, \dots, s_{k-1+j})} \quad (3)$$

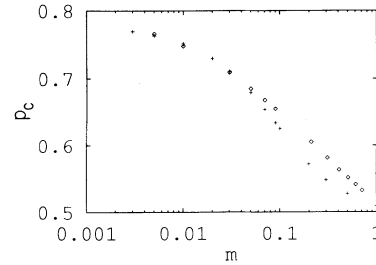


FIG. 2. Crossover of the critical transition point  $p_c(m)$  as the function of nonlocal site-exchange strength ( $m$ ). The crossover is continuous and  $p_c$  tends to mean-field approximation value (0.5). Data from extrapolation to multipoint approximation (+) and simulation ( $\diamond$ ).

expect that the universality changes toward the mean-field solution, as correlations are destroyed by mixing.

The stochastic, nonlocal site-exchange CA rule defined by two subrules applied alternatively. The first subrule is a synchronous one :

In principle,  $2^n - 1$  parameters are required to define the probability of all the  $n$ -point configurations. This number, however, is drastically reduced by the following conditions: In the stationary state the particle distribution is assumed to be symmetric with respect to translation and reflection. Furthermore, the block probability consistency results:

$$P_n(s_1, \dots, s_n) = \sum_{s_{n+1}} P_{n+1}(s_1, \dots, s_n, s_{n+1}),$$

$$P_n(s_1, \dots, s_n) = \sum_{s_0} P_{n+1}(s_0, s_1, \dots, s_n).$$

Adding site exchange to the synchronous rule the concentration does not change but correlations are destroyed and the system is expected to move toward mean-field-like behavior.

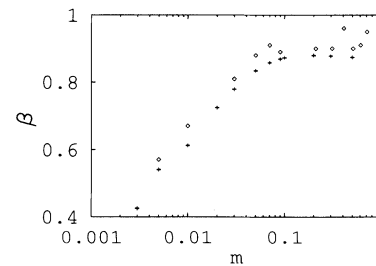


FIG. 3. Crossover of the critical  $\beta$  exponent vs  $m$ . The universality changes from that of directed percolation ( $\beta = 0.277$ ) of the mixing-free model ( $m = 0$ ) continuously to the universality characterized by  $\beta = 1$ . Data from extrapolation to multipoint approximation (+) and simulation ( $\diamond$ ).

Nonlocal site exchange can be taken into account in functions  $f$  by adding and subtracting terms to  $P_n$  block probabilities, describing configurations, which differ by particles added to or removed from a given configuration. For small  $m$  the mixing mechanism can be well described by single particle jumps in and out of an  $n$  block.

Larger  $m'$  ( $\gg m$ ) mixing—exhibiting multi-site-exchange—can be approximated by iterating these modifications on functions  $f$ ,  $m'/m$  times. We found that, choosing the single jump probability smaller than  $m < 0.01$ , the results remain the same within 4–5 digit accuracy.

By solving the set of nonlinear equations for a given level ( $n$ ) of approximation and fixed  $m$  we obtain  $c_n(p, m)$  order parameter curves showing different approximations to the phase transition (Fig. 1). For the  $n = 5$  level approximation we have a system of 13 nonlinear equations to be solved.

As we see the convergence to the simulated curve is slow—owing to the criticality—the critical point of the best (five-point level) approximation approaches the  $p_c = 0.8086$  value by 10% only for the  $m = 0$  mixing-free case. The  $p_c$  changes at every second level of approximation only. Between  $n = \text{odd}$  and  $n + 1$  the curvature of the  $c_n(p, m)$  curves increases. This means a better approximation to the order parameter exponent  $\beta$ .

An extrapolated curve fitted on the  $n = 1, 3, 5$  solutions by least square quadratic regression is also shown in Fig. 1. This calculation gives  $p_c = 0.78$ . The points of the limit curve are determined for each  $c$  value from the  $p_n(c, m) - 1/n$  data pairs as the intercept of the best quadratic fit. The application of simple quadratic regression seems to be a good approximation because even the linear fitting gives very small error. In comparison, our best result of six pair approximation with Padé extrapolation gave  $p_c = 0.7986$  and  $\beta = 0.29$  [5]. The pair approximation for the mixing-free model was possi-

ble since in the steady state the system can be built up from (01) and (00) blocks and one could treat them as new variables making the extended mean-field technique more powerful. Here we could not follow the same strategy because we want to take into account single particle jumps, which destroys the two-block structure.

We have determined the  $c_n(p, m)$  solutions for  $0 < m < 1$  and  $n = 1, \dots, 5$  levels. For each  $m$  an extrapolated  $c(p, m)$  limit curve ( $n \rightarrow \infty$ ) has been calculated by quadratic regression from the  $n = 1, 3, 5$  curves. The  $p_c(m)$  and  $\beta(m)$  are determined simultaneously from the extrapolated curves by least square fitting [4].

Figures 2 and 3 show the  $p_c(m)$  and  $\beta(m)$  results compared with simulations [9]. Both  $p_c(m)$  and  $\beta(m)$  tend to the mean-field values smoothly when increasing  $m$ . The transition remains second order. We can observe the change of universality from the DP class to a universality characterized by the  $\beta = 1$  mean-field exponent. The continuous crossover shows that the nonlocal site exchange—where a particle can jump to any distance—does not imply an infinite “effective interaction length” for this CA contrary to what was assumed for another dynamical system, i.e., for the kinetic Ising model [12,13].

In summary, the extended mean-field technique has been applied to stochastic CA with site exchange. Approximate solution up to the five-point level has been evaluated. Following quadratic extrapolation the limit curves of order parameter show how the transition depends on the mixing strength  $m$ . The transition remains second order, but interpolates between the directed percolation and a mean-field-like universality. The results agree well with simulation.

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